The number of solutions of an equation related to a product of multilinear polynomials

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Abstract

We look at the number of solutions of an equation of the form $f_1f_2...f_k = a$ in a finite field, where each f_i is a multilinear polynomial. We use two methods to construct a solution of this problem for the cases $a = 0$, $a \neq 0$, and we generally get a semi-explicit formula. We show that this formula can generate a more efficient algorithm than the traditional algorithm which consists to make a systematic computation. We also give explicit formulas in some special cases, and an application of our main result to the search of the weight hierarchy of the multilinear code with separated variables.

Keywords

Linear systems, multilinear polynomials, finite fields, separated variables, exponential sums.

MSC Codes

Primary classification: 11T06; Secondary classification: 11T23, 11T71.

I INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with q elements, $(m,n,k) \in (\mathbb{N}-\{0\})^3$ with $m \leq n$, $\{J_1, J_2, ..., J_m\}$ a partition of $\{1,...,n\}$, $(f,a) \in \mathbb{F}_q[X_1,..,X_n] \times \mathbb{F}_q$, A an $k \times m$ matrix with entries in $\mathbb{F}_q$. We assume that f satisfies the following:

(1.1) $$f = f_1f_2...f_k,$$

where :

(1.2) $$f_i(X_1,..,X_n) = \sum_{j=1}^{m} a_{ij} \prod_{\tau \in J_j} X_\tau\ ,\quad A = (a_{ij})_{\substack{1\leq i\leq k\\ 1\leq j\leq m}}.$$

The polynomials f_i are called multilinear polynomials, ([3] p. 32). In this present paper, the problem of determining the number of solutions in $\mathbb{F}_q^n$ of the following equation is considered:

(1.3) $$f(X_1,..,X_n) = a.$$

The solution of this problem is known only for special cases of (1.1) ([1], [2], [3]). We determine for the general case a semi-explicit formula for the solution of this problem. We obtain an explicit formula in special cases. When $a \neq 0$, the considered method is based on a generalization of the following formula:

$$N\left(h\left(X\right)g\left(X\right)=a\right)=\sum_{u\in\mathbb{F}_q^*}\left(N\left(h\left(X\right)=u\ \&\ g\left(X\right)=\tfrac{a}{u}\right)\right), \tag{1.4}$$

where $N\left(h\left(X\right)=u\ \&\ g\left(X\right)=a/u\right)$ denotes the number of common solutions of the equations $h\left(X\right)=u$ and $g\left(X\right)=a/u$.

When $a=0$, the considered method is based on the so-called " inclusion - exclusion principle". Note that if $m\geq 2$ and if $a_{ij}\neq 0$ for at least two entries j, the polynomials of the form (1.2) are strictly contained in the class of irreducible polynomials. Then, it could be interesting in a futur study to determine a sufficient condition to have the form (1.1). When this form exists, there exist algorithms to find these irreducible factors [4]. In our study, we suppose that the associated irreducible factors to f are known and our aim is to determine an expression for the number of solutions of the equation (1.3).

The organisation of this paper is the following: In section II, we continue the presentation of the considered notations, we recall basic results concerning linear systems and exponential sums. The "inclusion - exclusion principle" is also recalled. We compute formulas for the numbers of solutions of special systems of equations in Section III. These formulas will be useful in the treatment of the considered problem. The special case $k\in\{1,2\}$ is considered in this Section.

The main result for the general case is Proposition IV.1. In section V, we apply the obtained result to two "numerical" examples such that $k\in\{2,3\}$, $a_{ij}\in\{0,1\}$, $m\in\{3,4\}$, $n\in\{7,8\}$. We show from these examples that the obtained formula can generate a clearly more efficient algorithm than the "traditional" one which consists to make a systematic computation of all the solutions of this equation.

In section VI we show that the research of the number of zeros of products $f=f_1f_2...f_k$ of the form (1.1) is important when we compute the weight hierarchy of a multilinear code. To conclude in Section VII, we give three examples of problems that can be investigated from the present study.

II NOTATIONS, BASIC RESULTS

Recall that $m\leq n$. We assume:

$$k\leq m, \tag{2.1}$$

$$A \text{ is an } k\times m \text{ matrix of rank of } k. \tag{2.2}$$

Let $\quad 1\leq i_1<i_2<...<i_l\leq k,\ \ (b_1,...,b_l)\in\mathbb{F}_q^l.$

The system:

$$\left\{\begin{array}{l} f_{i_1}\left(X_1,..,X_n\right)=b_1,\\ \\ f_{i_2}\left(X_1,..,X_n\right)=b_2,\\ \qquad\quad .\\ \qquad\quad .\\ f_{i_l}\left(X_1,..,X_n\right)=b_l,\end{array}\right. \tag{2.3}$$

is equivalent to:

$$A_{[i_1,i_2,...,i_l]}\left(\prod_{\tau\in J_1} X_\tau, \prod_{\tau\in J_2} X_\tau, ..., \prod_{\tau\in J_m} X_\tau\right)^T = (b_1, ..., b_l)^T,$$

where

$$A_{[i_1,i_2,...,i_l]} = (a_{i_t,j})_{\substack{1\le t\le l\\ 1\le j\le m}}.$$

We can write:

$$(2.4a)\qquad B_{[i_1,i_2,...,i_l]}\left(\prod_{\tau\in J_{\nu_1}} X_\tau, \prod_{\tau\in J_{\nu_2}} X_\tau, ..., \prod_{\tau\in J_{\nu_l}} X_\tau\right)^T = (b_1, ..., b_l)^T - C_{[i_1,i_2,...,i_l]}\left(\prod_{\tau\in J_{\nu'_1}} X_\tau, \prod_{\tau\in J_{\nu'_2}} X_\tau, ..., \prod_{\tau\in J_{\nu'_{m-l}}} X_\tau\right)^T,$$

where

$$(2.4b)\qquad B_{[i_1,i_2,...,i_l]} = \left(a_{i_t,j}\right)_{\substack{1\le t\le l\\ j\in\{\nu_1,...,\nu_l\}\subset\{1,...,m\}}}$$

is an invertible $l\times l$ submatrix of $A_{[i_1,i_2,...,i_l]}$,

$$(2.4c)\qquad C_{[i_1,i_2,...,i_l]} = \left(a_{i_t,j}\right)_{\substack{1\le t\le l\\ j\in\{1,...,m\}-\{\nu_1,...,\nu_l\}}}$$

is an $l\times(m-l)$ submatrix of A,

$$(2.4d)\qquad \left\{\nu'_1, ..., \nu'_{m-l}\right\} = \{1, ..., m\} - \{\nu_{1,...,}\nu_l\}.$$

We will use the following notations:

$$(2.5a)\qquad \begin{cases} B^{-1}_{[i_1,i_2,...,i_l]} = \left(a'_{i_t,j}\right)_{\substack{t\in\{1,...,l\}\\ j\in\{\nu_{1,...,}\nu_l\}\subset\{1,...,m\}}}, \\ B^{-1}_{[i_1,i_2,...,i_l]}C_{[i_1,i_2,...,i_l]} = \left(\sigma^{t,j}_{[i_1,i_2,...,i_l]}\right)_{\substack{1\le t\le l\\ 1\le j\le m-l}}, \\ I_{\nu'} = J_{\nu'_1}\cup J_{\nu'_2}\cup .. \cup J_{\nu'_{m-l}}. \end{cases}$$

We denote by

$$(2.5b)\qquad N\left(f_{i_1}, ..., f_{i_l}, b_1, ..., b_l\right),$$

the number of solutions in $\mathbb{F}_q^n$ of the system of equations (2.3). The number of solutions in $\mathbb{F}_q^n$ of the equation (1.3) is denoted by

$$(2.5c)\qquad N\left(f, a\right).$$

To treat the considered problem, we use a method based on the following relations:

$$(2.6a)\qquad N(f,a)=\sum_{\substack{(a_1,...,a_k)\in\mathbb{F}_q^k\\ a_1....a_k=a}} N(f_1,...,f_k,a_1,...,a_k)\ ,$$

when $a\neq 0$,

$$(2.6b)\qquad N(f,0)=N(f_1,0)+N(f_2,0)+...+N(f_k,0)-\sum_{1\leq i_1<i_2\leq k} N(f_{i_1},f_{i_2},0,0)$$

$$+\sum_{1\leq i_1<i_2<i_3\leq k} N(f_{i_1},f_{i_2},f_{i_3},0,0,0)+....+(-1)^{j+1}\sum_{1\leq i_1<i_2..<i_j\leq k} N\left(f_{i_1},f_{i_2},...,f_{i_j},0,0,...,0\right)$$

$$+...+(-1)^{k+1}N(f_1,f_2,...,f_k,0,0,...,0)\ ).$$

The considered method in [2] to solve this problem in the particular case $k=1$, $a=0$, is based on exponential sums. We shall use a similar method in Section III to solve the more general case $k=1$.
For all $u\in\mathbb{F}_q$, the following function is an additive character on $\mathbb{F}_q$

$$(2.7)\qquad \Psi_u\ (v)=\exp\left(\tfrac{2i\pi}{p}Tr_{\mathbb{F}_q}(uv)\right),$$

where $Tr_{\mathbb{F}_q}(uv)$ is the absolute trace of uv.

We will use the following basic results:

$$(2.8)\qquad \sum_{u\in\mathbb{F}_q}\Psi_u\ (v)=\begin{cases}0 & \text{if}\ \ v\neq 0,\\ q & \text{if}\ \ v=0.\end{cases}$$

$$(2.9)\qquad \sum_{v\in\mathbb{F}_q}\Psi_u\ (v)=\begin{cases}0 & \text{if}\ \ u\neq 0,\\ q & \text{if}\ \ u=0.\end{cases}$$

We also need the following result given in [2]:

PROPOSITION II-1

Let $(\alpha,u,d)\in(\mathbb{F}_q-\{0\})^2\times(\mathbb{N}-\{0\})$, *then*:

$$(2.10)\qquad \sum_{(X_1,X_2,..,X_d)\in\mathbb{F}_q^d}\Psi_u\left(\alpha\prod_{i\in\{1,..,d\}}X_i\right)=q^d-q\,(q-1)^{d-1}.$$

We conclude this present Section by recalling the wellknown map [1]:

$$(2.11)\qquad \begin{array}{l}\kappa:\mathbb{F}_q\rightarrow\{-1,q-1\}\,,\\ \kappa(X)=\begin{cases}-1 & \text{if}\ \ X\neq 0,\\ q-1 & \text{if}\ \ X=0.\end{cases}\end{array}$$

III NUMBER OF SOLUTIONS OF SPECIAL SYSTEMS

In the following Proposition, we determine the solution of the problem for the special case $k=1$.

PROPOSITION III-1

Assume that $k=1$ *in* (1.1), *then*:

$$(3.1) \qquad N\left(f,a\right)=q^{n-1}+\kappa\left(a\right)q^{n-1}\left[\prod_{\{l/a_{1l}\neq 0\}}\left(1-\left(\frac{q-1}{q}\right)^{|J_l|-1}\right)\right],$$

$$(Recall\ \ that\ N\left(f,a\right)\ \ is\ the\ number\ of\ \ solutions\ of\ \ (1.3)).$$

Proof

First, (2.8) and (2.9) imply that:

$$N\left(f,a\right)=\frac{1}{q}\left(\sum_{(X_1,X_2,..,X_n)\in\mathbb{F}_q^n}\sum_{u\in\mathbb{F}_q}\Psi_u\left(f\left(X_1,X_2,..,X_n\right)-a\right)\right),$$

thus:

$$\begin{aligned}
N\left(f,a\right)= &\ \frac{1}{q}\left(\sum_{u\in\mathbb{F}_q}\sum_{(X_1,X_2,..,X_n)\in\mathbb{F}_q^n}\Psi_u\left(f\left(X_1,X_2,..,X_n\right)-a\right)\right)\\
&=\frac{1}{q}\left(\sum_{u\in\mathbb{F}_q}(\Psi_u\left(-a\right)\sum_{(X_1,X_2,..,X_n)\in\mathbb{F}_q^n}\Psi_u\left(f\left(X_1,X_2,..,X_n\right)\right))\right)\\
&=\frac{1}{q}\left(\sum_{u\in\mathbb{F}_q}(\Psi_u\left(-a\right)\sum_{(X_1,X_2,..,X_n)\in\mathbb{F}_q^n}\Psi_u(\sum_{j=1}^{m}a_{1j}\prod_{\tau\in J_j}X_\tau))\right)\\
&=\frac{1}{q}(\sum_{u\in\mathbb{F}_q}[\Psi_u\left(-a\right)\prod_{\{k/a_{1k}\neq 0\}}\sum_{\left(X_1,X_2,..,X_{|J_k|}\right)\in\mathbb{F}_q^{|J_k|}}\Psi_u(a_{1k}\prod_{\tau\in J_k}X_\tau)]).
\end{aligned}$$

After, from Proposition II-1 we get:

$$\begin{aligned}
N\left(f_1,a\right)&=q^{n-1}+q^{n-1-\sum_{\{k/a_{1k}\neq 0\}}|J_k|}\left(\sum_{u\in\mathbb{F}_q^*}[\Psi_u\left(-a\right)\prod_{\{k/a_{1k}\neq 0\}}\left(q^{|J_k|}-q\left(q-1\right)^{|J_k|-1}\right)\right)\\
&=q^{n-1}+q^{n-1}[\prod_{\{k/a_{1k}\neq 0\}}\left(1-\left(\frac{q-1}{q}\right)^{|J_k|-1}\right)][\sum_{u\in\mathbb{F}_q^*}\Psi_u\left(-a\right)].
\end{aligned}$$

From (2.8) and (2.11) we deduce $\sum_{u\in\mathbb{F}_q^*}\Psi_u(-a)=\kappa\left(a\right)$, thus

$$N\left(f,\ a\right)=q^{n-1}+\kappa\left(a\right)q^{n-1}\prod_{\{l/a_{1l}\neq 0\}}\left(1-\left(\frac{q-1}{q}\right)^{|J_l|-1}\right),$$

which is the desired result.■

The following Corollary which is a consequence of Proposition III-1, will be useful for instance in the proof of Proposition III-3.

COROLLARY III-2

Let $(w,\alpha,d)\in\mathbb{F}_q\times(\mathbb{F}_q-\{0\})\times(\mathbb{N}-\{0\})$, *then the number* $N\left(\alpha\prod_{\tau=1}^{d}X_\tau,w\right)$ *of solutions in* $\mathbb{F}_q^d$ *of the equation:*

$$\alpha\prod_{\tau=1}^{d}X_\tau=w$$

is

$$N\left(\alpha\prod_{\tau=1}^{d}X_\tau,w\right)=q^{d-1}+\kappa\left(w\right)q^{d-1}\left(1-\left(\frac{q-1}{q}\right)^{d-1}\right).$$

PROPOSITION III-3

Let $1\leq i_1<i_2<...<i_l\leq k$, $(b_1,...,b_l)\in\mathbb{F}_q^l$, *then*

(3.2) $N\left(f_{i_1},f_{i_2},...,f_{i_l},b_1,b_2,...,b_l\right)=$

$$\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{\left|I_{\nu'}\right|}}\prod_{t=1}^{l}\left(q^{\left|J_{\nu_t}\right|-1}+\kappa\left(\sum_{j=1}^{l}a'_{i_t,j}b_j-\sum_{j=1}^{m-l}\sigma^{t,j}_{[i_1,i_2,...,i_l]}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)q^{\left|J_{\nu_t}\right|-1}\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{\nu_t}\right|-1}\right)\right),$$

$$\left(I_{\nu'},\ J_{\nu'_t},\ a'_{i_t,j},\ \sigma^{t,j}_{[i_1,i_2,...,i_l]}\ \textit{are defined in}\ (2.4b),(2.4c),(2.4d),(2.5a)\right).$$

Proof

First recall that the matrix A is of rank k, then the existence of the $l\times l$ invertible matrix $B_{[i_1,i_2,...,i_l]}$ introduced in $(2.4a)$, $(2.4b)$ is justified. Consequently, the equality (2.4) is equivalent to:

$$\left(\prod_{\tau\in J_{\nu_1}}X_\tau,\prod_{\tau\in J_{\nu_2}}X_\tau,....,\prod_{\tau\in J_{\nu_l}}X_\tau\right)^T=B^{-1}_{[i_1,i_2,...,i_l]}\left(b_1,...,b_l\right)^T-$$

$$B^{-1}_{[i_1,i_2,...,i_l]}C_{[i_1,i_2,...,i_l]}\left(\prod_{\tau\in J_{\nu'_1}}X_\tau,\prod_{\tau\in J_{\nu'_2}}X_\tau,....,\prod_{\tau\in J_{\nu'_{m-l}}}X_\tau\right)^T,$$

with $\left\{\nu_{1,...,}\nu_l\right\}\cup\left\{\nu'_1,...,\nu'_{m-l}\right\}=\left\{1,...,m\right\}$, (see $(2.4d)$).

It follows that:

(3.2) $$\prod_{\tau\in J_{\nu_t}}X_\tau=\sum_{j=1}^{l}a'_{i_t,j}b_j-\sum_{j=1}^{m-l}\sigma^{t,j}_{[i_1,i_2,...,i_l]}\prod_{\tau\in J_{\nu'_j}}X_\tau\quad,\quad t=1,...,l,$$

applying Corollary III-2 to (3.2), we get:

$N\left(f_{i_1},f_{i_2},...,f_{i_j},b_1,b_2,...,b_l\right)=$

$$\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{\left|I_{\nu'}\right|}}\prod_{t=1}^{l}\left(q^{\left|J_{\nu_t}\right|-1}+\kappa\left(\sum_{j=1}^{l}a'_{i_{tt},j}b_j-\sum_{j=1}^{m-l}\sigma^{t,j}_{[i_1,i_2,...,i_l]}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)q^{\left|J_{\nu_t}\right|-1}\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{\nu_t}\right|-1}\right)\right),$$

which is the announced result.■

Now in the following Proposition, we give a semi-explicit formula for the number of solutions of (1.3) for the case $k=2$.

PROPOSITION III-4

Assume that $k=2$ *in* (1.1). *With the notations* $(2.5a)$, $(2.5b)$ *and* $(2.5c)$, *we have*:

$$N(f,0)=2q^{n-1}+ (q-1)\,q^{n-1}\left(\prod_{\{l/a_{1l}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)+\prod_{\{l/a_{2l}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right)$$
$$-\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{\left|I_{\nu'}\right|}}\prod_{t=1}^{2}\left(q^{\left|J_{\nu_t}\right|-1}+\kappa\left(\sum_{j=1}^{m-2}\sigma_{[1,2]}^{t,j}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)q^{\left|J_{\nu_t}\right|-1}\left(1-\left(\tfrac{q-1}{q}\right)^{\left|J_{\nu_t}\right|-1}\right)\right),$$

$$N(f,a)=$$
$$\sum_{u\in\mathbb{F}_q^*}\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{\left|I_{\nu'}\right|}}\prod_{t=1}^{2}q^{|J_{\nu t}|-1}\left(1+\kappa\left(a'_{1,1}u+\frac{a'_{1,2}}{u}a-\sum_{j=1}^{m-2}\sigma_{[1,2]}^{t,j}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)\left(1-\left(\tfrac{q-1}{q}\right)^{\left|J_{\nu_t}\right|-1}\right)\right),$$
$$if \quad a\neq 0.$$

Proof

First consider the case $a=0$. Then, from $(2.6b)$ we have:

$$N(f,0)=N(f_1,0)+N(f_2,0)-N(f_1,f_2,0,0).$$

Using (3.1) we get:

(3.3) $N(f_1,0)+N(f_2,0)=$

$$q^{n-1}+\kappa(0)\,q^{n-1}\left[\prod_{\{l/a_{1l}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right]+q^{n-1}+\kappa(0)\,q^{n-1}\left[\prod_{\{l/a_{2l}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right]$$
$$=2q^{n-1}+(q-1)\,q^{n-1}\left[\prod_{\{l/a_{1l}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)+\prod_{\{l/a_{2l}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right].$$

After, using Proposition III-3 in the case $(l,b_1,b_2)=(2,0,0)$, we can write:

$$N(f_1,f_2,0,0)=\sum_{(X_i)_{i\in I_{\nu'}}\in F_q^{\left|I_{\nu'}\right|}}\prod_{t=1}^{2}\left(q^{\left|J_{\nu_t}\right|-1}+\kappa\left(\sum_{j=1}^{m-2}\sigma_{[1,2]}^{t,j}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)q^{\left|J_{\nu_t}\right|-1}\left(1-\left(\tfrac{q-1}{q}\right)^{\left|J_{\nu_t}\right|-1}\right)\right),$$

the desired result follows in the case $a=0$.

Now assume that $a\neq 0$. From $(2.6a)$ we have:

$$N(f,a)=\sum_{u\in\mathbb{F}_q^*}N\left(f_1,f_2,u,\frac{a}{u}\right).$$

Let $u\in\mathbb{F}_q^*$, by Proposition III-3 in the case $(l,b_1,b_2)=\left(2,u,\frac{a}{u}\right)$ we get:

$N\left(f_1,f_2,u,\frac{a}{u}\right)=$

$$\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{|I_{\nu'}|}}\prod_{t=1}^{2}q^{|J_{\nu_t}|-1}\left(1+\kappa\left(a'_{1,1}u+\frac{a'_{1,2}}{u}a-\sum_{j=1}^{m-2}\sigma^{t,j}_{[1,2]}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{\nu_t}|-1}\right)\right),$$

consequently, we easily get the announced result in the case $a\neq 0$. ■

IV THE GENERAL PROBLEM

Here, we consider the equation (1.3), namely: $f(X_1,..,X_n)=a$, where $f=f_1f_2...f_k$ and $f_i(X_1,..,X_n)=\sum_{j=1}^{m}a_{ij}\prod_{\tau\in J_j}X_\tau$, and we assume that the rank of the matrix $A=(a_{ij})_{1\le i\le k,1\le j\le m}$ is k. In the general case, the number of solutions $N(f,a)$ of $f(X_1,..,X_n)=a$ is given by next Proposition:

PROPOSITION IV-1

With the notations (2.5a),

$$N(f,0)=kq^{n-1}+(q-1)q^{n-1}\left(\sum_{i=1}^{k}\prod_{\{l/a_{il}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right)+$$

$$\sum_{l=2}^{k}(-1)^{l+1}\sum_{1\le i_1<i_2<..<i_l\le k}\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{|I_{\nu'}|}}\prod_{t=1}^{l}q^{|J_{\nu_t}|-1}\left(1+\kappa\left(\sum_{j=1}^{m-l}\sigma^{t,j}_{[i_1,i_2,...,i_l]}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{\nu_t}|-1}\right)\right),$$

$$N(f,a)=$$

$$\sum_{\substack{(a_1,...,a_k)\in\mathbb{F}_q^k\\ a_1a_2..a_k=a}}\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{|I_{\nu'}|}}\prod_{t=1}^{k}q^{|J_{\nu_t}|-1}\left(1+\kappa\left(\sum_{j=1}^{k}a'_{t,j}a_j-\sum_{j=1}^{m-k}\sigma^{t,j}_{[1,...,k]}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{\nu_t}|-1}\right)\right),$$

if $a\neq 0$.

Proof

First consider the case $a=0$. From Proposition III-1 and for all $i\in\{1,..,k\}$:

$$N(f_i,0)=q^{n-1}+(q-1)q^{n-1}\left[\prod_{\{l/a_{il}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right],$$

thus:

$$\sum_{i=1}^{k}N(f_i,0)=kq^{n-1}+(q-1)q^{n-1}\sum_{i=1}^{k}\left[\prod_{\{l/a_{il}\neq 0\}}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_l|-1}\right)\right].$$

Now let $1\le i_1<i_2<...<i_l\le k$, Proposition III-3 gives:

$$N(f_{i_1},f_{i_2},...,f_{i_l},0,0,...,0)=$$

$$\sum_{(X_i)_{i\in I_{\nu'}}\in\mathbb{F}_q^{|I_{\nu'}|}}\prod_{t=1}^{l}\left(q^{|J_{\nu_t}|-1}+\kappa\left(\sum_{j=1}^{m-t}\sigma^{t,j}_{[i_1,i_2,...,i_l]}\prod_{\tau\in J_{\nu'_j}}X_\tau\right)q^{|J_{\nu_t}|-1}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{\nu_t}|-1}\right)\right).$$

After, we get the desired result for the case $a = 0$ from $(2.6b)$.

Now assume that $a \neq 0$ and let $(a_1, ..., a_k) \in \mathbb{F}_q^k$ such that $a_1 a_2 ... a_k = a$, then from Proposition III-3, we can write:

$N(f_1, f_2, ..., f_k, a_1, ..., a_k) =$

$$\sum_{(X_i)_{i \in I_{\nu'}} \in \mathbb{F}_q^{|I_{\nu'}|}} \prod_{t=1}^{k} \left(q^{|J_{\nu_t}|-1} + \kappa \left(\sum_{j=1}^{k} a'_{t,j} a_j - \sum_{j=1}^{m-k} \sigma_{[1,2,...,k]}^{t,j} \prod_{\tau \in J_{\nu'_j}} X_\tau \right) q^{|J_{\nu_i}|-1} \left(1 - \left(\frac{q-1}{q} \right)^{|J_{\nu_i}|-1} \right) \right),$$

it follows from $(2.6a)$ that:

$N(f, a) =$

$$\sum_{\substack{(a_1,...,a_k) \in \mathbb{F}_q^k \\ a_1 ... a_k = a}} \sum_{(X_i)_{i \in I_{\nu'}} \in \mathbb{F}_q^{|I_{\nu'}|}} \prod_{t=1}^{k} (q^{|J_{\nu_t}|-1} \left(1 + \kappa \left(\sum_{j=1}^{k} a'_{t,j} a_j - \sum_{j=1}^{m-k} \sigma_{[1,2,...,k]}^{t,j} \prod_{\tau \in J_{\nu'_j}} X_\tau \right) \left(1 - \left(\frac{q-1}{q} \right)^{|J_{\nu_t}|-1} \right) \right)),$$

which completes the proof.■

The formula given in previous Proposition is generally a semi-explicit formula because of terms

$$\kappa \left(\sum_{j=1}^{m-t} \sigma_{[i_1,i_2,...,i_l]}^{t,j} \prod_{\tau \in J_{\nu'_j}} X_\tau \right), \quad \kappa \left(\sum_{j=1}^{k} a'_{t,j} a_j - \sum_{j=1}^{m-k} \sigma_{[1,2,...,k]}^{t,j} \prod_{\tau \in J_{\nu'_j}} X_\tau \right).$$

In the next Proposition, particular cases related to the matrix A are considered.

PROPOSITION IV-2

If $m = k$ *and* $a \neq 0$ *then*:

(4.1) $N(f, a) =$

$$\sum_{(a_1,..,a_{m-1}) \in (\mathbb{F}_q^*)^{m-1}} [\prod_{i=1}^{m} \left(q^{|J_i|-1} + \kappa \left(\sum_{j=1}^{m-1} a'_{ij} a_j + a'_{im} \frac{a}{a_1 .... a_{m-1}} \right) q^{|J_i|-1} \left(1 - \left(\frac{q-1}{q} \right)^{|J_i|-1} \right) \right)].$$

If

(4.2) $$\begin{cases} m = 2k, \\ A = (D_1 \ D_2), \end{cases}$$

where D_1 *and* D_2 *are two invertible diagonale* $k \times k$ *matrices, then*:

(4.3) $N(f, a) =$

$$(q-1)^{k-1} \prod_{j=1}^{k} \left[q^{|J_i|+|J_{k+i}|-1} - q^{|J_i|+|J_{k+i}|-1} \left(1 - \left(\frac{q-1}{q} \right)^{|J_i|-1} \right) \left(1 - \left(\frac{q-1}{q} \right)^{|J_{k+i}|-1} \right) \right],$$

if $a \neq 0$,

(4.4) $\quad N(f,0) = kq^{n-1} + (q-1)\,q^{n-1}\sum_{i=1}^{k}[\left(1-\left(\frac{q-1}{q}\right)^{|J_i|-1}\right)\left(1-\left(\frac{q-1}{q}\right)^{|J_{k+i}|-1}\right)]-$

$$\sum_{1\le i_1<i_2\le k} q^{n-\sum_{j=1}^{2}(\left|J_{i_j}\right|+\left|J_{k+i_j}\right|)} \prod_{j=1}^{2}(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|-1}\left(1+\kappa(0)\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{i_j}\right|-1}\right)\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{k+i_j}\right|-1}\right)\right))+$$

$...+$

$$(-1)^{l+1}\sum_{1\le i_1<..<i_l\le k} q^{n-\sum_{j=1}^{l}(\left|J_{i_j}\right|+\left|J_{k+i_j}\right|)} \prod_{j=1}^{l}(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|-1}\left(1+\kappa(0)\left(1\text{-}\left(\frac{q-1}{q}\right)^{\left|J_{i_j}\right|-1}\right)\left(1\text{-}\left(\frac{q-1}{q}\right)^{\left|J_{k+i_j}\right|-1}\right)\right))$$

$$+...+(-1)^{k+1}\prod_{j=1}^{k}(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|-1}\left(1+\kappa(0)\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{i_j}\right|-1}\right)\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{k+i_j}\right|-1}\right)\right)).$$

Proof

First consider the case $m=k$. Then A is an invertible $m\times m$ matrix.
Let $(a_1,a_2,...,a_{m-1})\in\left(\mathbb{F}_q^*\right)^{m-1}$. In $(2.4a)$, $(2.5a)$ we can take:

$$B_{[i_1,i_2,...,i_l]} = A,\ C_{[i_1,i_2,...,i_l]} = 0,\ (b_1,...,b_m) = \left(a_1,a_2,...,a_{m-1},\frac{a}{a_1a_2....a_{m-1}}\right),$$

$$l=m,\ [i_1,i_2,...,i_m] = [1,2,..,m]\,,\ I_{\nu'}=\emptyset,$$

then from (3.2), we can write:

$$N\left(f_1,f_2,...,f_m,a_1,....,a_{m-1},\frac{a}{a_1....a_{m-1}}\right) =$$

$$\prod_{i=1}^{m}\left(q^{|J_i|-1}+\kappa\left(\sum_{j=1}^{m-1}a'_{i,j}a_j + a'_{im}\frac{a}{a_1...a_{m-1}}\right)q^{|J_i|-1}\left(1-\left(\frac{q-1}{q}\right)^{|J_i|-1}\right)\right)$$

and the result (4.1) follows from $(2.6a)$.

Now consider the case (4.2) and let $(a_1,a_2,...,a_{k-1})\in\left(\mathbb{F}_q^*\right)^{k-1}$. Recalling that $A=(a_{ij})$, it's easy to see that: $D_1 = Diag\,(a_{11},...,a_{kk})$, $D_2 = Diag\,(a_{1,k+1},a_{2,k+2}...,a_{k,k+k})$, $a_{ij}=0$ for all (i,j) such that $j-i\notin\{0,i\}$, $a_{ii}a_{i,k+i}\neq 0$ for all $i\in\{1,...,k\}$. This means that we can make the following choices in $(2.4a)$:

$$B_{[i_1,i_2,...,i_l]} = D_1,\ C_{[i_1,i_2,...,i_l]} = D_2,\ (b_1,...,b_k)^T = \left(a_1,a_2,...,a_{k-1},\frac{a}{a_1a_2....a_{k-1}}\right)^T,$$

$$l=k,\ [i_1,i_2,...,i_k] = [1,2,..,k]\,,\ (\nu_1,..,\nu_k) = (1,2,..,k)\,,$$

$$\left(\nu'_1,..,\nu'_{m-k}\right) = (k+1,..,m-k)\,,$$

thus:

$$\left(\prod_{\tau\in J_1}X_\tau,\prod_{\tau\in J_2}X_\tau,....,\prod_{\tau\in J_k}X_\tau\right)^T = D_1^{-1}\left(a_1,...,a_{k-1},\frac{a}{a_1a_2....a_{k-1}}\right)^T -$$

$$D_1^{-1}D_2\left(\prod_{\tau\in J_{k+1}}X_\tau,\prod_{\tau\in J_{k+2}}X_\tau,....,\prod_{\tau\in J_m}X_\tau\right)^T$$

and for all $i\in\{1,..,k\}$,

(4.5) $$\prod_{\tau\in J_i} X_\tau = a_{ii}^{-1}a_i - a_{ii}^{-1}a_{i,k+i}\prod_{\tau\in J_{k+i}} X_\tau,$$
with $a_k = \frac{a}{a_1....a_{k-1}}$.

We deduce from Proposition III-1 and the hypothesis $(a_{ii}, a_i, a_{i,k+i}) \in (\mathbb{F}_q^*)^3$, that the number of solutions in $\mathbb{F}_q^{|J_i|+|J_{k+i}|}$ of (4.5) is:

$$q^{|J_i|+|J_{k+i}|-1} - q^{|J_i|+|J_{k+i}|-1}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_i|-1}\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{k+i}|-1}\right).$$

Then, it's easy to see that:

$$N\left(f_1, f_2, ..., f_k, a_1, ..., a_{k-1}, \tfrac{a}{a_1a_2....a_{k-1}}\right) =$$
$$\prod_{i=1}^{k}\left[q^{|J_i|+|J_{k+i}|-1} - q^{|J_i|+|J_{k+i}|-1}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_i|-1}\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{k+i}|-1}\right)\right]$$

and from $(2.6a)$ the result (4.3) follows.

Now, let $1 \le i_1 < i_2 < ... < i_l \le k$. The system of equations (2.3) in the case (4.2) and $(b_1, ..., b_l) = (0, ..., 0)$, namely:

$$\begin{cases} f_{i_1}(X_1, .., X_n) = 0, \\ f_{i_2}(X_1, .., X_n) = 0, \\ \quad ... \\ f_{i_l}(X_1, .., X_n) = 0, \end{cases}$$

is equivalent to

$$\begin{cases} \prod_{\tau\in J_{i_1}} X_\tau + a_{i_1i_1}^{-1}a_{i_1,k+i_1}\prod_{\tau\in J_{k+i_1}} X_\tau = 0, \\ \prod_{\tau\in J_{i_2}} X_\tau + a_{i_2i_2}^{-1}a_{i_2,k+i_2}\prod_{\tau\in J_{k+i_2}} X_\tau = 0, \\ \quad ... \\ \prod_{\tau\in J_{i_l}} X_\tau + a_{i_li_l}^{-1}a_{i_l,k+i_l}\prod_{\tau\in J_{k+i_l}} X_\tau = 0, \end{cases}$$

thus:

$$N(\ f_{i_1}, f_{i_2}, ..., f_{i_l}, 0, 0, .., 0) =$$
$$q^{n-\sum_{j=1}^{l}(\left|J_{i_j}\right|+\left|J_{k+i_j}\right|)}\prod_{j=1}^{l}\left(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|-1} + \kappa(0)\,q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|-1}\left(1\text{-}\left(\tfrac{q-1}{q}\right)^{\left|J_{i_j}\right|-1}\right)\left(1-\left(\tfrac{q-1}{q}\right)^{\left|J_{k+i_j}\right|-1}\right)\right).$$

After by Proposition III-1, we have for all $j \in \{1, .., l\}$:

$$N(f_i, 0) = q^{n-1} + \kappa(0)\,q^{n-1}\left(1-\left(\tfrac{q-1}{q}\right)^{|J_i|-1}\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{k+i}|-1}\right),$$

consequently from $(2.6b)$, we have:

$$N(f, 0) = kq^{n-1} + \kappa(0)\,q^{n-1}\sum_{i=1}^{k}\left[\left(1-\left(\tfrac{q-1}{q}\right)^{|J_i|-1}\right)\left(1-\left(\tfrac{q-1}{q}\right)^{|J_{k+i}|-1}\right)\right] -$$

$$\sum_{1\leq i_1<i_2\leq k} q^{n-\sum_{j=1}^{2}(\left|J_{i_j}\right|+\left|J_{k+i_j}\right|)}\left[\prod_{j=1}^{2}(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|\text{-}1}\left(1+\kappa(0)\left(1\text{-}\left(\frac{q-1}{q}\right)^{\left|J_{i_j}\right|\text{-}1}\right)\left(1\text{-}\left(\frac{q-1}{q}\right)^{\left|J_{k+i_j}\right|\text{-}1}\right)\right))\right]+...+$$

$$(\text{-}1)^{l+1}\sum_{1\leq i_1<...<i_l\leq k} q^{n\text{-}\sum_{j=1}^{l}(\left|J_{i_j}\right|+\left|J_{k+i_j}\right|)}\prod_{j=1}^{l}(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|\text{-}1}\left(1+\kappa(0)\left(1\text{-}\left(\frac{q\text{-}1}{q}\right)^{\left|J_{i_j}\right|\text{-}1}\right)\left(1\text{-}\left(\frac{q\text{-}1}{q}\right)^{\left|J_{k+i_j}\right|\text{-}1}\right)\right))$$

$$+...+(-1)^{k+1}\prod_{j=1}^{k}(q^{\left|J_{i_j}\right|+\left|J_{k+i_j}\right|-1}\left(1+\kappa(0)\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{i_j}\right|-1}\right)\left(1-\left(\frac{q-1}{q}\right)^{\left|J_{k+i_j}\right|-1}\right)\right)),$$

which is the formula (4.4) .■

This Proposition indicates for instance that for particular interesting cases related to the matrix A, we can get an explicit formula for the number of solutions of (1.3) .

V NUMERICAL EXAMPLES

V-1 A FIRST EXAMPLE

Consider the following particular situation of (1.3):

(5.1) $$(X_1X_2+X_5X_6X_7)(X_3X_4+X_5X_6X_7)=a\ .$$

Then:

$$\begin{cases} (k,m,n)=(2,3,7)\,, \\ A=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}, \\ J_1=\{1,2\}\,,\ J_2=\{3,4\}\,,\ J_3=\{5,6,7\}\,. \end{cases}$$

We have the following Proposition.

PROPOSITION V-1

Let α be a primitive element of $\mathbb{F}_q$, $1\leq\tau\leq\frac{q-1}{2}$. Then the number of solutions of (5.1) is:

(5.2) $$N(f,0)=2q^6+3q^5-13q^4+16q^3-9q^2+2q,$$

(5.3) $$N\left(f,\alpha^{2\tau}\right)=(q-1)^2q\left(q^3+q^2-2q+2\right),$$

(5.4) $$N\left(f,\alpha^{2\tau-1}\right)=(q-1)^3q\left(q^2+2q-2\right).$$

Proof

In (2.4a) we can take:

$l=2$, $[i_1,i_2]=[1,2]$, $(\nu_1,\nu_2)=(1,2)$, $\nu_1'=3$, $B_{[1,2]}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $C_{[1,2]}=\begin{pmatrix}1\\1\end{pmatrix}$,

thus:

$$B_{[1,2]}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix},\ B_{[1,2]}^{-1}\ C_{[1,2]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma_{[1,2]}^{1,1} \\ \sigma_{[1,2]}^{2,1} \end{pmatrix}.$$

From the above assumptions and Proposition III-4, we first can establish easily (5.2) and secondly we can deduce that for all $1 < t \leq q-1$:

$$(5.5)\ N\left(f,\ \alpha^{t}\right) = \sum_{u\in\mathbb{F}_q^*}\ \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(\frac{\alpha^t}{u} - X_5X_6X_7\right)\right)\right],$$

it follows that:

$$\begin{aligned} N\left(f,\ \alpha^{2\tau}\right) = & \sum_{u\in\{-\alpha^\tau,\alpha^\tau\}}\ \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(\frac{\alpha^{2\tau}}{u} - X_5X_6X_7\right)\right)\right] + \\ & \sum_{u\notin\{-\alpha^\tau,0,\ \alpha^\tau\}}\ \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(\frac{\alpha^{2\tau}}{u} - X_5X_6X_7\right)\right)\right] \\ = & T_1 + T_2, \end{aligned}$$

where:

$$(5.6)\qquad T_1 = \sum_{u\in\{-\alpha^\tau,\alpha^\tau\}}\ \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(\frac{\alpha^{2\tau}}{u} - X_5X_6X_7\right)\right)\right],$$

$$(5.7)\qquad T_2 = \sum_{u\notin\{-\alpha^\tau,0,\ \alpha^\tau\}}\ \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(\frac{\alpha^{2\tau}}{u} - X_5X_6X_7\right)\right)\right],$$

thus:

$$\begin{aligned} T_1 = & \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left(q+\kappa\left(-\alpha^\tau - X_5X_6X_7\right)\right)^2 + \sum_{(X_5,X_6,X_7)\in\mathbb{F}_q^3} \left(q+\kappa\left(\alpha^\tau - X_5X_6X_7\right)\right)^2 \\ = & \sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7=-\alpha^\tau}} \left(q+\kappa\left(0\right)\right)^2 + \sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7\neq-\alpha^\tau}} \left(q+\kappa\left(-\alpha^\tau - X_5X_6X_7\right)\right)^2 \\ & + \sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7=\alpha^\tau}} \left(q+\kappa\left(0\right)\right)^2 + \sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7\neq\alpha^\tau}} \left(q+\kappa\left(\alpha^\tau - X_5X_6X_7\right)\right)^2. \end{aligned}$$

Using Corollary III-2, we get:

$$\begin{aligned} (5.8)\qquad T_1 &= 2\left(q-1\right)^2\left(2q-1\right)^2 + 2\left(q^3 - \left(q-1\right)^2\right)\left(q-1\right)^2 \\ &= 2\left(q-1\right)^2 q\left[q^2+3q-2\right]. \end{aligned}$$

Now, (5.7) implies that:

$$\begin{aligned} T_2 = & \sum_{u\notin\{-\alpha^\tau,0,\ \alpha^\tau\}} [\sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7=u}} \left[\left(q+\kappa(0)\right)\left(q+\kappa\left(\frac{\alpha^{2\tau}}{u} - X_5X_6X_7\right)\right)\right] + \\ & \sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7=\frac{\alpha^{2\tau}}{u}}} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(0\right)\right)\right] + \\ & \sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7\notin\left\{u,\frac{\alpha^{2\tau}}{u}\right\}}} \left[\left(q+\kappa\left(u - X_5X_6X_7\right)\right)\left(q+\kappa\left(\frac{\alpha^{2\tau}}{u} - X_5X_6X_7\right)\right)\right]]. \end{aligned}$$

It follows from Corollary III-2 that:

$$(5.9)\quad T_2 = \sum_{u \notin \{-\alpha^{\tau}, 0,\ \alpha^{\tau}\}} \left[2(q-1)^3(2q-1) + \left(q^3 - 2(q-1)^2\right)(q-1)^2\right]$$
$$= (q-3)\left[2(q-1)^3(2q-1) + \left(q^3 - 2(q-1)^2\right)(q-1)^2\right]$$
$$= q(q-3)(q-1)^2\left(q^2 + 2q - 2\right),$$

and from (5.8) and (5.9), the formula (5.3) follows.

Now a similar analysis leads to the following:

$$N\left(f,\ \alpha^{2\tau-1}\right) = \sum_{u \in \mathbb{F}_q^*} [\sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7=u}} \left[(q+\kappa(0))\left(q+\kappa\left(\tfrac{\alpha^{2\tau-1}}{u} - X_5X_6X_7\right)\right)\right] +$$
$$\sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7=\frac{\alpha^{2\tau-1}}{u}}} \left[(q+\kappa(u - X_5X_6X_7))(q+\kappa(0))\right] +$$
$$\sum_{\substack{(X_5,X_6,X_7)\in\mathbb{F}_q^3/ \\ X_5X_6X_7\notin\left\{u,\frac{\alpha^{2\tau-1}}{u}\right\}}} \left[(q+\kappa(u - X_5X_6X_7))\left(q+\kappa\left(\tfrac{\alpha^{2\tau-1}}{u} - X_5X_6X_7\right)\right)\right],$$
$$= (q-1)\left[2(q-1)^3(2q-1) + \left(q^3 - 2(q-1)^2\right)(q-1)^2\right].$$

It's follows that $N\left(f,\ \alpha^{2\tau-1}\right) = (q-1)^3 q\left(q^2+2q-2\right)$, which is the formula (5.4) .■

The result given in this Proposition indicates for particular cases of (1.3) that the obtained solution is a simple explicit formula.

V-2 A SECOND EXAMPLE

The example that we consider here shows us that the obtained formula for $N(f,a)$ in Proposition IV-1 is more efficient than a trivial systematic computation of all the solutions. In the semi-explicit formula (5.12), the use of a classical machine is certainly required, but lead to easier computation involving sums and the obvious function κ. This example is the following:

$$(5.10)\qquad (X_1 + X_6X_7X_8)(X_1 + X_2X_3 + X_6X_7X_8)(X_2X_3 + X_4X_5) = a$$

We have:

$$\begin{cases} (k,m,n) = (3,4,8), \\ A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\ J_1 = \{1\},\ J_2 = \{2,3\},\ J_3 = \{4,5\},\ J_4 = \{6,7,8\} \end{cases}$$

and the following result:

PROPOSITION V-2

The number $N(f,a)$ of solutions of (5.10) *is*:

(5.11) $$N(f,0) = 3q^7 - 3q^6 + q^3\left(2q^2 - 2q + 1\right),$$

(5.12) $$N(f,a) = q^3 \sum_{a_1,a_2\in\mathbb{F}_q^*} \left(q + \kappa\left(a_2 - a_1\right)\right)\left(q + \kappa\left(a_1 - a_2 + \frac{a}{a_1a_2}\right)\right),$$

if $a \neq 0$.

Proof

With obvious notations, Proposition IV-1 gives:

(5.13) $$N(f,0) = \xi - \sum_{1\leq i_1<i_2\leq 3} C_{i_1,i_2} + C_{1,2,3}.$$

It is easily seen that $\xi = 3q^7 + (q-1)q^5$. To compute $C_{1,2}$, we consider in (2.4a):

$$B_{[1,2]} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C_{[1,2]} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Hence $B_{[1,2]}^{-1}\, C_{[1,2]} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left(\sigma_{[1,2]}^{i,j}\right)_{i,j}$ and $C_{1,2} = q^5\left(2q-1\right)$, as is easy to check. Likewise we obtain: $C_{1,3} = C_{2,3} = q^6 + q^3 \sum_{(X_4,X_5)\in F_q^2} \kappa\left(X_4X_5\right) = q^6 + q^4\left(q-1\right)$ and $C_{1,2,3} = \left(2q-1\right)^2 q^3$.

Then it is easy to see that (5.13) is equivalent to (5.11).

Now assume that $a \neq 0$. According to proposition IV-1 and (2.4a), we have:

$$N(f,a) = \sum_{a_1,a_2\in\mathbb{F}_q^*} \sum_{X_6,X_7,X_8} \prod_{t=1}^{3} q^{\left|J_{\nu_t}\right|-1} \left(1 + \kappa\left(\sum_{j=1}^{3} a'_{t,j}a_j - \sigma_{[1,2,3]}^{t,1} X_6X_7X_8\right)\left(1 - \left(\frac{q-1}{q}\right)^{\left|J_{\nu_t}\right|-1}\right)\right),$$

$$B_{[1,2,3]}^{-1} = \left(a'_{t,j}\right)_{t,j} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad B_{[1,2,3]}^{-1} C_{[1,2,3]} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left(\sigma_{[1,2,3]}^{i,1}\right)_{i,1}.$$

A straightforward computation gives us formula (5.12).■

VI WEIGHT HIERARCHY OF A CLASS OF CODES

Let us denote by $\mathcal{J}$ the partition $\{J_1, J_2, ..., J_m\}$ of $\{1, ..., n\}$. Following Cherdieu and Rolland in [2], we can introduce the map

$$\begin{array}{cccc} c: & E(q,n,\mathcal{J}) & \to & \mathbb{F}_q^{q^n} \\ & f & \mapsto & \left(f(x)\right)_{x\in\mathbb{F}_q^n} \end{array}$$

where $E(q,n,\mathcal{J})$ is the set of all multilinear polynomials with separated variables of the form $f(X_1, .., X_n) = \sum_{j=1}^{m} a_j \prod_{\tau\in J_j} X_\tau$ where $a_1, ..., a_m \in \mathbb{F}_q$. The map c is injective and its image $\operatorname{Im} g$ is known as a multilinear code with separated variables. We write $\operatorname{Im} g = C(q,n,\mathcal{J})$. Proposition III.1 gives:

$$N(f,a) = q^{n-1} + (q-1)\,q^{n-1}\left[\prod_{\{j/a_j\neq 0\}}\left(1-\left(\frac{q-1}{q}\right)^{|J_j|-1}\right)\right],$$

and we easily deduce:

PROPOSITION VI-1
The lenght of the code $C(q,n,\mathcal{J})$ is q^n, and its dimension is m. The weight $\omega(c(f))$ of the code-word $c(f)$, where $f(X_1,..,X_n) = \sum_{j=1}^{m} a_j \prod_{\tau\in J_j} X_\tau$, is

$$\omega(c(f)) = q^{n-1}(q-1)\left[1-\prod_{\{j/a_j\neq 0\}}\left(1-\left(\frac{q-1}{q}\right)^{|J_j|-1}\right)\right],$$

and the minimum distance of $C(q,n,\mathcal{J})$ is

$$\operatorname{dist}(C(q,n,\mathcal{J})) = q^{n-s}(q-1)^s$$

where $s = \operatorname{Max}_{1\leq j\leq m}(|J_j|)$.

If C denotes a code with parameters $[n,r,d]$, and if D denotes one of its subcodes, we recall that $\chi(D) = \{i\in\{1,...,n\}\ /\ \exists\,(x_1,\ldots,x_n)\in D \text{ with } x_i\neq 0\}$ is the support of D, and that the number $\omega(D) = |\chi(D)|$ of elements in $\chi(D)$ is the Wei weight of D ([6] p. 1412). The h^{th} minimum weight of C is given by

$$d_h = d_h(C) = \operatorname{Min}\{\omega(D)\ /\ D \text{ subcode of } C \text{ with } \dim(D)=h\}.$$

The minimum distance of C is d_1, and the weight hierarchy of the code C is the sequence $\{d_1,\ldots,d_r\}$.

If $D = \operatorname{Vect}(f_1,...,f_h)$ denotes the subcode of $C(q,n,\mathcal{J})$ generated by the polynomials f_1, ..., f_h in $E(q,n,\mathcal{J})$, and if we write $\mathbb{F}_q^n = \{P_1,...,P_{q^n}\}$, the Wei weight of D is

$$\begin{aligned}\omega(D) &= |\{i\in\{1,...,q^n\}\ /\ \exists g\in D\quad g(P_i)\neq 0\}| \\ &= \left|\left\{i\in\{1,...,q^n\}\ /\ \exists\lambda_1,...,\lambda_h\in\mathbb{F}_q\quad \sum_{j=1}^{h}\lambda_j f_j(P_i)\neq 0\right\}\right| \\ &= |\{i\in\{1,...,q^n\}\ /\ \exists j\in\{1,...,h\}\qquad f_j(P_i)\neq 0\}|,\end{aligned}$$

thus

$$\omega(D) = q^n - N(f_1,...,f_h,0,...,0).$$

Each polynomial f_i $(1\leq i\leq h)$ can be written in the form $f_i(X_1,..,X_n) = \sum_{j=1}^{m} a_{ij}\prod_{\tau\in J_j} X_\tau$ and the rank of the matrix $A=(a_{ij})$ is k as soon as we assume that D is of dimension k. We can therefore apply Proposition IV.1 to compute $\omega(D)$ and d_h.

VII CONCLUSION

As we have seen, Proposition IV.1 gives us a semi-explicit formula to compute the number $N(f_1,...,f_k,0,...,0)$ when the rank of the matrix A is k, and it would be interesting to build an explicit program to do it, and compute the weight hierarchy of the code $C(q,n,\mathcal{J})$. This could be a numerical continuation of this work.

Several problems can arise from the present paper, and we gives here two ways for further investigations. First, it would be interesting to define a partition $\{\triangle_1, \triangle_2, ..., \triangle_\eta\}$ of $\mathbb{F}_q^n$ and write the number $N(f,a)$ in this way:

$$N(f,a) = \sum_{j=1}^{\eta} N_{\triangle_j}(f,a) = \sum_{j=1}^{\eta} \sum_{a_1 ... a_k = a} N_{\triangle_j}(f_1, ..., f_k, a_1, ..., a_k),$$

where $N_{\triangle_j}(f,a)$ denotes the number of solutions of (1.3) in $\triangle_j$, and $N_{\triangle_j}(f_1, ..., f_k, a_1, ..., a_k)$ stands for the number of solutions of the system $f_i(X_1, ..., X_n) = a_i$ $(1 \leq i \leq k)$ in $\triangle_j$. Specific partitions could lead us to less calculus. For instance,

$$\triangle_1 = \left\{(u_1, ..., u_n) \in \mathbb{F}_q^n \,/\, \exists\, (i_1, i_2, .., i_m) \in J_1 \times J_2 \times ... \times J_m \quad (u_{i_1}, ..., u_{i_m}) = 0\right\},$$
$$\triangle_2 = \left\{(u_1, ..., u_n) \in \left(\mathbb{F}_q^*\right)^n \,/\, \forall \tau \in \{1, .., m\} \quad \forall\, (i,j) \in J_\tau \times J_\tau \quad u_i = u_j \text{ for all } \right\},$$

lead to a trivial numbers and diagonal equations.

The second problem consists to study the more general case where:

$$f = f_1 f_2 ... f_k,$$
$$f_i(X_1, ..., X_n) = \sum_{j=1}^{s} a_{ij} \prod_{\tau \in J_j^{(i)}} X_\tau, \; i = 1, .., k,$$
$$\left\{J_1^{(1)}, J_2^{(1)}, ..., J_s^{(1)}\right\}, \left\{J_1^{(2)}, J_2^{(2)}, ..., J_s^{(2)}\right\}, ..., \left\{J_1^{(k)}, J_2^{(k)}, ..., J_s^{(k)}\right\} \text{ are partitions of } \{1, ..., n\},$$

and to obtain semi-explicit formulas for the number of solutions of $f(X_1, ..., X_n) = a$ in this case.

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